## January 2003


| | | | |
|---|---|---|---|
| **LEAP** | **ALTAIR-Western Region USA** | **DYNAMAX** | **Cril Technology** |
| **GISSETA** | **MFAC** | **DYNA*more*** | **FLOTREND** |
| **KOSTECH** | **ERAB** | **THEME** | **ALTAIR-Italy** |
| **CAD-FEM** | **Prof Gennaro Monacelli** | **Dr. David Benson** | **Dr. Alexey I. Borovkov** |
| **Dr. Ted Belytschko** | **Dr. Taylan Altan** | **Dr. Bhavin V. Mehta** | **Prof. Ala Tabiei** |

# FEA Information Inc. Worldwide News

## January - Volume 4

## Issue 01-2003

# New Functionality in LS-PREPOST 1.0

1. **Full LS-DYNA 970 keyword support and keyword entities creation.**
    a. LS-PREPOST 1.0 is now able to read/write and edit all LS-DYNA 970 keywords in a form-based user interface.
    b. Keyword entities are easily generated, deleted and manipulated through a graphical interface. For example: *SET data is easily handled by part, by element and by other techniques.
2. **A Subsystem concept is introduced when importing multiple keyword models**
    a. The subsystem concept gives the user better control over the imported models.
    b. Subsystem model entity ids are automatically offset for a user when more than one keyword is imported into a model.
    c. Each Include file is maintained as a separate subsystem.
3. **Extensive mesh manipulation features are now available.**
    a. Translation, Rotation, Scaling, Projection, Reflection and Transformation for nodes and elements.
    b. Normal consistency is checked for parts with shell elements and auto reverse with a seed element defined.
    c. Shell element quality checking.
    d. A fast element deletion capability.
4. **Metal forming related features**
    a. Creation of new parts by offsetting elements along the element normal or moving and copying all or part of the original part.
    b. Separation measurements between parts can be displayed as fringe plots. This is supported in the pre and post-processing modes.
    c. Part traveled to part contact calculation is implemented.
5. **Occupant Positioning – Improved capabilities**
    a. Multiple occupants can be imported into one single model. The user interface allows one to switch between the two occupants models.
    b. The configuration file can be setup to automatically load an occupant model upon startup.
    c. All keyword items for the occupants such as local coordinates, joints, and GCs are now positioned along with the limbs.
    d. Manipulation of the occupant model is now recorded in angle files which can be reloaded for future use.
6. **201 Head Impact Positioning**
    a. After positioning a Head impact model the user can generate an LS-DYNA 970 input file.
    b. The configuration file can be modified to have the head impact model loaded automatically at startup.
7. **Airbag Folding capability**
    a. Flat bag folding with thin, thick, tuck and spiral fold.
    b. The folding sequence can be animated
8. **SPH element generation**
    a. Users now can generate SPH parts in simple geometries such as boxes, spheres and cylinders.
    b. Materials models for SPH simulation are built into an internal database for users to setup the properties for the generated SPH part.

**ANSYS "Forensic" Engineering Studies (c) Copyright ANSYS Inc.**
**Thiokol, FAA, Weidlinger Associates**
**Adapted from *www.ansys.com***

This case study briefly examines the application of ANSYS software in the postmortem examinations of three prominent catastrophic events. While there may not have been a way to predict or prevent these tragic incidents in the first place, it is worthy to note the significant role which ANSYS software plays in helping engineers comprehend the unique mechanics of design failure under extreme circumstances.

## Introduction:

Although designers and engineers work diligently to ensure the viability and safety of their designs before they are built, sometimes, unforeseen circumstances can occur. Whether they are the result of mechanical failure, natural phenomena or acts of sabotage, each year these incidents cause billions of dollars worth of damage and significant loss of human life to even the best engineered designs, if not their utter destruction

When these catastrophes occur, the United States Federal Emergency Management Agency (FEMA) and other government agencies often enlist the services of engineers charged to ascertain precisely what went wrong in an attempt to prevent a future recurrence. More often in these cases, when engineers need to utilize simulation software as a "forensic" investigation tool, the solution they most often turn to comes from ANSYS Inc.

## The Destruction of the Space Shuttle Challenger - January 28, 1986

Seventy-three seconds after launching from the Kennedy Space Center in Florida, an ignition of mixed liquid oxygen and hydrogen fuel, brought about as the result of a faulty engine sealant, destroyed the shuttle orbiter Challenger.

Then-president Ronald Reagan appointed a special commission to investigate the cause of the accident and develop corrective measures. Among the members of the investigation team were engineers from Thiokol Space Operations of Brigham City, Utah (U.S.A.)-the original designers of the solid rocket boosters (SRB), where the failure was believed to have occurred.

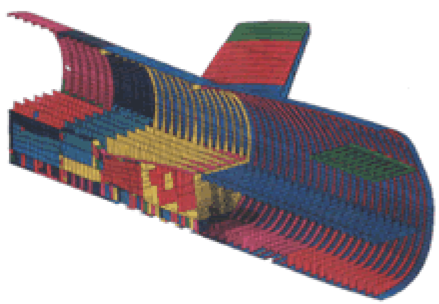
Utilizing ANSYS' non-linear analysis capabilities, the Thiokol team was able to identify that unusually cold weather that day had caused the rubber O-rings-which seal the components of the SRB together-to stiffen. Subsequently, the SRB lost cohesion causing a chain reaction that resulted in the shuttle's destruction.

"At the time," observes Troy Stratten, principle structural analyst at Thiokol, "this was probably the largest ANSYS non-linear model ever run." Indeed, the representation of the tang and clevis joint region (left), where additional high stress concentrations were detected, contained 30,000 elements and 100,000 degrees of freedom.

Once these design flaws were identified, NASA required the SRB to be re-certified for operation. The results generated by the ANSYS simulation allowed Thiokol's engineers to redesign the SRB's joint system to minimize gapping and characterize stress concentrations. ANSYS proved to be the key in ensuring the safety and success of future shuttle missions.

## The Crash of TWA Flight 800 - July 19, 1996

Fourteen minutes after taking off from New York City's John F. Kennedy International Airport, a Boeing 747-100-identified as Trans World Airlines Flight 800-exploded and crashed into the sea nine miles off Long Island, New York (U.S.A.).



To this day, the precise cause of the crash has not been identified conclusively. However, a joint investigation spearheaded by the Boeing Company, in cooperation with the National Transportation Safety Board (NTSB) and the Federal Aviation Administration (FAA), yielded a "most likely" scenario to explain the events of that fateful night.

A thorough inspection of the recovered wreckage-confirmed via simulations performed in ANSYS-led investigators to hypothesize that the fuel vapor/air mixture inside the nearly empty center wing tank (CWT) was ignited by elevated temperatures. The ignition of this combination of gases ruptured the fuel tank which, in turn, caused the plane to violently break up. While fuel tank fires or explosions of this type are rare, two other previously documented occurrences confirm that they fall completely within the realm of possibility.

Patrick Safarian, then senior specialist engineer for Boeing, recalls the key to satisfactorily resolving this important issue was found in the robust structural and Computational Fluid Dynamics (CFD) capabilities in ANSYS. "Performing failure analysis at this level was, until then, probably unheard of," he enthuses. "It may still be unsurpassed."

Safarian's ANSYS rendering of Flight 800 (above) was based upon an ANSYS-modified finite element model of a 747-400 freighter, consisting of 120,000 shell and beam elements with 750,000 degrees of freedom. He observes that while the initial source of ignition was unknown, "We moved (it) around, remodeled, re-analyzed and took careful note of the results. This gave us a good degree of confidence in the failure sequences."

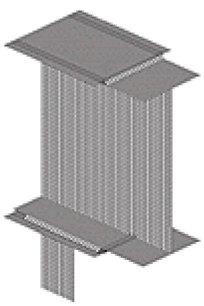
As a result of Safarian's efforts, the NTSB advised the FAA to take numerous steps to prevent a recurrence. These include thorough examinations of the physical condition of fuel tanks and all related components on more than 850 aircraft currently in use throughout the world. Other recommended precautions-such as pumping inert gases into fuel tanks, refueling from ground tanks (which store fuel at a lower temperature level), and carrying an "appropriate" amount of fuel in tanks at all times (as full tanks are less likely to explode than empty ones)-are also under consideration.

**The Collapse of the World Trade Center - September 11, 2001**

While catastrophes that occur via natural or mechanical means are difficult enough to foresee, it is nearly impossible to predict those that result from a deliberate attack by an unknown enemy.
Following these events, FEMA formed a coalition with the Structural Engineering Institute of the American Society of Civil Engineers (SEI/ASCE)-as well as the city of New York and several other Federal agencies and professional organizations-to document the performance of buildings at ground zero. Their goal was to document the sequence of events, likely root causes, and methods or technologies that may improve or mitigate the observed building performance.

Although WTC buildings 1, 2 and 7 collapsed completely, other structures such as the Bankers Trust Building (located at 130 Liberty Street) remained standing, sustaining only moderate localized damage. Robert Smilowitz, consulting engineer with Weidlinger Associates Applied Science division in New York City, led the team studying the Bankers Trust Building.

While not originally designed to sustain the loss of a column over a significant portion of its height, this structure's ability to arrest collapse demonstrated an inherent tenacity in the moment-connected steel frame lattice. To gain a better understanding of the building's response to the impact of debris from the falling WTC structures and identify specific design features that contributed to this performance, Smilowitz's team utilized simulation software from ANSYS Inc.



In order to represent the structural behavior in the damaged state, the team had to develop non-linear spring representations of the girder/column moment connections (left). Detailed plate models of the connections were developed and analyzed parametrically to determine the appropriate non-linear spring characteristics. These properties were then specified at the corresponding connections in the ANSYS model of the building

.

Thanks in large part to ANSYS' static non-linear analysis capabilities, the team was able to determine the diminished capacity of the connections resulted from the "out-of-plane" bending associated with the damaged state of the structure. This partially explains the damage pattern, which was contained in the northeast face of the building, extending from the initial impact area on the 22nd floor down to the eighth floor.

"It is difficult to draw conclusions," observes Smilowitz in his report to FEMA. "More detailed study is required to understand how the collapse was halted." He believes that a complete FEA analysis on the Bankers Trust Building-conducted in ANSYS-will aid current and future builders in constructing buildings better able to avert catastrophic collapses in the event of abnormal loading conditions.

**SGI® Decisions Support Center: Information Dominance Solution For the 21st Century**

**Article: http://www.sgi.com/features/2002/dec/dsc/index.html**

Today, governments worldwide are inundated with an abundance of data but a shortage of information that supports the decision-making process. In the future, a variety of sources and sensors will continue to generate ever-increasing amounts of such data. That data includes text-based information and records along with more complex media types such as video, audio, imagery, scans, electronic emissions, and other geo-referenced data.

To deal with this data overload problem, users must be able to manage, correlate, fuse, and visualize data to provide enhanced time to insight and to help government decision makers "see the threat" for applications including homeland security, crisis management, emergency training and preparedness, and command and control. Human visual perception is dominated by the sense of sight, so the ability to visually represent data is the key to turning data into information.

Command-and-control systems provide military commanders with a real-time view of the environment in which their forces operate. The SGI DSC solution enables military commanders to know where their forces are, what they are doing, what assets are available and how to bring them to bear on the mission

SGI has developed a variety of visualization and computing technologies ideally suited to allow government decision makers to rapidly assimilate the growing amounts and diverse types of data being collected. Finding the few important bits of information out of mountains of data is the job of the SGI Decision Support Center (DSC) solution, which uses large-scale visualization, high-performance computing, and the management of complex data to provide mission-critical information to support rapid and confident decision-making cycles.

A DSC acts as a data fusion engine that lays out massive amounts of vital information in a real-time virtual visual panorama to help decision makers see the big picture and focus on making the right decisions. In order for something to be called a decision support center, it must improve the decision-making process. Improvements can be measured in time (making faster decisions), quality (making better, more informed decisions), and confidence (making the right decisions).

DSCs increase the value of collected data by allowing multisource data fusion and presentation in an immersive environment where government decision makers can see the entire problem using an intuitive interface. These applications require significant graphics power to display the data in real time. They also require complex data management and high-performance computing to process and fuse the data into a visual representation.

DSCs, such as this mobile command center, make possible faster and more informed decisions. Globally integrated decision making is now possible. DSCs are well suited to support reach back capabilities, allowing collaboration up and down the chain of command in real time.

In the area of homeland security, SGI is working with its partners to develop a solution for decision support and communications called the Nuclear, Biological, and Chemical Threat Operation and Training Center (NBCOTC). The NBCOTC is a unique decision-making tool that combines computing, graphics, and display technology and is designed to provide faster time to insight. The NBCOTC is being designed to provide a command center for a sustained response to a large-scale crisis, as well as to provide a training environment to assure preparedness. Based on SGI® Reality Center™ products and technology, the NBCOTC will collect and fuse 2D and 3D data from multiple sources and enable decision makers to analyze, predict, and review actions for rehearsal and operations

DSCs are a natural outgrowth of the SGI Reality Center business. Close to 600 Reality Center facilities have been installed worldwide and are used for a broad range of applications including oil and gas exploration, drug research, and virtual prototyping of automobiles. In the commercial market, Reality Center facilities are used to support a variety of decision-making processes, including which car to build, where to drill an oil well, and what molecule is required to attack a virus. Now, that same technology is being applied to government challenges.

A recent example is Australia's New South Wales State Rail Authority, which is the first transport organization in the world to open a state-of-the-art virtual reality and simulation center that will enhance training and education levels for all State Rail employees. Using an SGI Reality Center facility, employees--including train drivers, guards, and station managers--will experience realistic scenarios covering all aspects of driver training, platform safety, emergency procedures training, customer service, and engineering maintenance.

A typical Reality Center facility in the commercial world is a theater environment with usually one point of control--one person, in the back of the room, controlling a graphics supercomputer with a single keyboard and mouse. In a DSC, there can be a fairly large number of direct participants (from 5 to 100) in the decision-making process. The majority of these participants are specialists contributing specific talents or skills to the center. Many of these specialists use special-purpose workstations or consoles that perform unique tasks, such as radar processing, weather forecasting, traffic control, or logistics.

DSCs such as this command tent, make possible faster and more informed decisions. Globally integrated decisions making is now possible. DSCs are well suited to support reachback capabilities, enabling remote personnel to access data wherever the battlespace or DSC is located

In the government market, DSCs are Reality Center environments that are used to collect data and analyze, predict, rehearse, operate, and review actions for exercises and operations. Huge amounts of complex data--imagery, signal, terrain, GPS and more--are fused into a large-scale display that provides decision makers with a complete view of the situation and all its variables in real time. Using that real-time data, decision makers can collaborate to find the best solutions during a crisis or contingency operation

SGI and its partners are working on a Crisis Management Center solution that would provide situational awareness and a common operating picture for the following natural and manmade disasters: earthquakes, hurricanes, storm surges, wildfire, oil/toxic spills, and debris removal. An example is a recently opened advanced visualization center at the Scripps Institution of Oceanography at the University of California, San Diego. Two universities, together with the California Institute for Telecommunications and Information Technology, are creating a prototype for collaborative scientific analysis that could also be used as a command-and-control facility for crisis management.

DSCs make possible faster and more informed decisions--and they defy geographical boundaries. Globally integrated decision making is now possible. These centers are well suited to support reach-back capabilities--enabling remote personnel to access data wherever the DSC is located--and allow collaboration up and down the chain of command in real time. So, regardless of its location, a DSC is a place for groups to make decisions quickly, collaboratively, and with confidence. SGI Decision Support Center solutions provide information dominance in an increasingly complex world.

## Highlights of News Pages Posted on FEA Information in December
## Archived on the News Page

| | |
|---|---|
| **Dec 02** | **ANSYS FEMXplorer** |
| | **LS-DYNA** Material Model Information |
| | **MFAC's** builds clusters of PCs running under Red Hat Linux |
| **Dec. 09** | **JRI - JMAG** a magnetic field analysis program |
| | **MSC.DYTRAN** for simulating the high-speed response of solids, structures, fluids |
| | **DYNAMAX –** Distributor located near Detroit |
| **Dec 23** | **OASYS –** Bra Analysis – structural performance |
| | **HP** Workstation ZX6000 |
| | **Theme Engineering** - Distributor in Korea |
| **Dec 30** | **Intel**® - Itanium® 2 processor |
| | **CEI** - EnSight Gold |
| | **Cril Technology Simulation –** Distributor in France |
| | |

## FEA Information Participants

| Headquarters | Company | |
|---|---|---|
| Australia | Leading Engineering Analysis Providers | www.leapaust.com.au |
| Canada | Metal Forming Analysis Corp. | www.mfac.com |
| France | Cril Technology Simulation | www.criltechnology.com |
| Germany | DYNAmore | www.dynamore.de |
| Germany | CAD-FEM | www.cadfem.de |
| India | GissEta | www.gisseta.com |
| Italy | Altair Engineering srl | www.altairtorino.it |
| Japan | The Japan Research Institute, Ltd | www.jri.co.jp |
| Japan | Fujitsu Ltd. | www.fujitsu.com |
| Korea | THEME Engineering | www.lsdyna.co.kr |
| Korea | Korean Simulation Technologies | www.kostech.co.kr |
| Russia | State Unitary Enterprise - STRELA | www.ls-dynarussia.com |
| Sweden | Engineering Research AB | www.erab.se |
| Taiwan | Flotrend Corporation | www.flotrend.com |
| UK | OASYS, Ltd | www.arup.com/dyna |
| USA | INTEL | www.intel.com |
| USA | Livermore Software Technology | www.lstc.com |
| USA | Engineering Technology Associates | www.eta.com |
| USA | ANSYS, Inc | www.ansys.com |
| USA | Hewlett Packard | www.hp.com |
| USA | SGI | www.sgi.com |
| USA | MSC.Software | www.mscsoftware.com |
| USA | DYNAMAX | www.dynamax-inc.com |
| USA | CEI | www.ceintl.com |
| USA | AMD | www.amd.com |
| USA | Dr. T. Belytschko | Northwestern University |
| USA | Dr. D. Benson | Univ. California – San Diego |
| USA | Dr. Bhavin V. Mehta | Ohio University |
| USA | Dr. Taylan Altan | The Ohio State U – ERC/NSM |
| USA | Prof. Ala Tabiei | University of Cincinnati |
| Russia | Dr. Alexey I. Borovkov | St. Petersburg State Tech. University |
| Italy | Prof. Gennaro Monacelli | Prode – Elasis & Univ. of Napoli, Federico II |

| 2003 | EVENTS PAGE LISTINGS ON FEA INFORMATION |
|---|---|
| Feb 18 | **Fujitsu LS-DYNA** Seminar at Makuhari System Laboratory |
| Feb 20-21 | 2nd LS-DYNA Users Conference by **GissETA India** Private Limited - dev@gisseta.com |
| March 18-19 | **Russian Automotive Conference** taking place in Moscow |
| May 19-21 | **BETECH 2003** taking place at the Hyatt Regency Dearborn hotel in Detroit, USA - **15th International Conference on Boundary Element Technology** |
| May 22 - 23 | **4th European LS-DYNA Conference** will be held in ULM, Germany presented by DYNAmore (Germany), Cril Technology Simulation (France), ARUP (United Kingdom), Engineering Research AB (Sweden) and STRELA (Russia) Call for Papers & Registration - (PDF 472KB) |
| June 3-5 | **Testing Expo 2003**, Stuttgart, Germany. A world's leading automotive test and evaluation exhibition & conference |
| June 17-20 | **The Second M.I.T. Conference on Computational Fluid and Solid Mechanics,** taking place at Massachusetts Institute of Technology Cambridge, MA.,USA The mission of the M.I.T. Conference is: "To bring together Industry and Academia and To nurture the next generation in computational mechanics" |
| Oct 29-31 | Hosted at the conveniently located Novi Expo Center in Detroit, Michigan**, Testing Expo North America 2003** will bring together, under one roof, leading test equipment manufacturers and test service providers. |

# Element Locking

David J. Benson

January 21, 2003

Under some circumstances the displacements calculated by the finite element method are orders of magnitude smaller than they should be, and when this happens, the elements are said to be *locking*. The two most common types of locking are *shear* and *pressure* locking. Locking occurs in lower order elements because an element's kinematics aren't rich enough to represent the correct solution. Shear locking occurs when elements are subjected to bending, and pressure locking occurs when the material is incompressible. Most of the research on reducing locking is devoted to elements with linear shape functions, with the remainder devoted to quadratic elements.

## 1 A Pathological Case of Volume Locking in Triangular Elements

Consider triangle 1, in Figure 1, which is defined by nodes 1 and 2 on the $x$ axis, and node 3 on the $y$ axis. The area of the triangle is $(x_2 - x_2)y_3/2$, and it must remain constant if the triangle is incompressible. If nodes 1 and 2 are fixed, then $y_3$ must remain constant and $v_3 = 0$. The remaining degree of freedom is the horizontal displacement $u_3$. Similarly, for the triangle 2, defined by nodes 4, 5, and 6, the only remaining degree of freedom is the vertical displacement $v_6$.

Two triangles may be assembled into a quadrilateral region, see Figure 2. Since incompressibility for triangle 1 requires $v_4 = 0$ and incompressibility for triangle 2 requires $u_4 = 0$, node 4 can't move, and the elements are completely locked up. With nodes 1 through 4 locked up, the nodes for triangles 3 and 4 will also be locked, as will the ones for triangles 5 and 6, see Figure 3. Again, since all previous triangles are locked, adding triangles

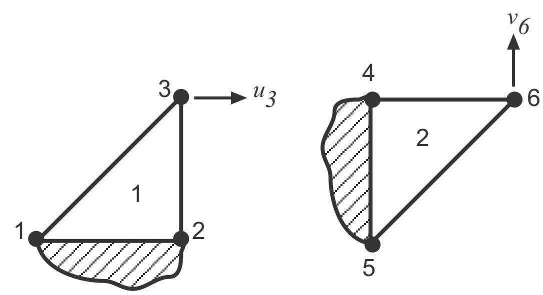



Figure 1: The remaining degrees of freedom for two incompressible constant strain triangles.

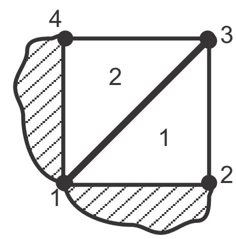



Figure 2: Two incompressible constant strain triangles assembled into a quadrilateral lock.

7 and 8 will result in their nodes also being locked. Elements can continue to be added in the same pattern, and all the nodes will be locked. Analogous problems occur in three dimensions with tetrahedral elements.

Locking is eliminated in this situation by using *crossed triangles* (Figure 4). Although the first quadrilateral generated by crossed triangles has only one degree of freedom, a large assembly of quadrilaterals generated by crossed triangles gives reasonable answers for incompressible problems. Crossed triangles, however, are still overly stiff in comparison to correctly formulated quadrilateral elements.

# 2 Locking in a Quadrilateral Element

Locking in linear quadrilateral elements occurs when they are subjected to bending. For simplicity, let's examine the simplest case of a quad in bending: The element is rectangular with a height $2h$ and a length $2b$. In the $x - y$ plane, the origin is located at the centroid of the element (see Figure 5).

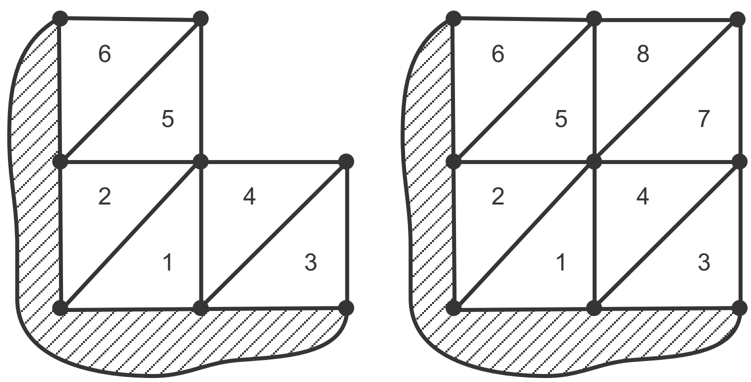



Figure 3: A sequence of triangular meshes that are completely locked.

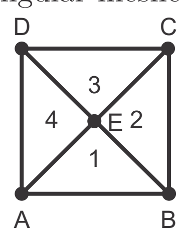



Figure 4: Crossed triangles don't lock like the previous configurations.

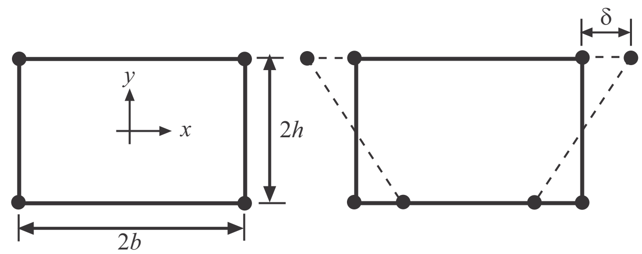



Figure 5: The geometry and kinematics of a quadrilateral element in a pure bending mode.

The neutral axis is the line $y = 0$, and cross-sections, which are initially perpendicular to the neutral axis, are defined by lines of constant $x$.

If the element is subjected to applied moments on the left and right edges, the resulting deformation is bilinear. As shown, it is symmetric about $x = 0$ and may be characterized by $\delta$, the displacement of node 3 in the $x$ direction.

$$u = \delta \left(\frac{x}{b}\right)\left(\frac{y}{h}\right) \tag{1}$$

$$v = 0. \tag{2}$$

Differentiating, the strain, as a function of $x$ and $y$, is

$$\epsilon_{11} = \frac{\delta y}{hb} \tag{3}$$

$$\epsilon_{22} = 0 \tag{4}$$

$$\epsilon_{12} = \frac{\delta x}{2hb}. \tag{5}$$

## 2.1 Shear Locking

Beam theory says that as the thickness of a beam approaches zero, the shear strain goes to zero, and all the load is carried by $\sigma_{11}$. However, the ratio of the shear strain to the tensile strain in the quadrilateral element is

$$\frac{\epsilon_{12}}{\epsilon_{11}} = \frac{x}{2y}. \tag{6}$$

The integration locations for $2 \times 2$ Gauss quadrature are $(\pm h/\sqrt{3}, \pm b/\sqrt{3})$ in the physical coordinate system. At the integration points, the ratio of the strains will therefore be proportional to $b/h$. Clearly, as $h/b$ approaches zero, the ratio $\epsilon_{12}/\epsilon_{11}$ approaches infinity, and the element locks up because the shear strain, and not the tensile strain, is carrying the load.

For a pure bending mode, the shear strain is zero along the line $x = 0$ according to Equation 5, and along the same line, $\epsilon_{11}$ is linear in $y$, which are the theoretical strains. Two-node beams which are formulated by degenerating solid elements (a common approach) therefore often use 1-point integration along the neutral axis to eliminate shear locking, and multi-point integration through the thickness to a accurately calculate the bending moments and force resultants.

## 2.2 Volume Locking

The volume strain is $\epsilon_{11} + \epsilon_{22} + \epsilon_{33}$. In our plane strain example, $\epsilon_{33}$ is always zero. An incompressible material in plane strain therefore requires $\epsilon_{22} = -\epsilon_{11}$. Looking at Equation 3, $\epsilon_{22}$ must be linear in $y$ just like $\epsilon_{11}$ to satisfy the incompressibility constraint. Therefore, with $v(0)$ being the displacement at $y = 0$,

$$\frac{\partial v}{\partial y} = -\frac{\delta y}{hb} \tag{7}$$

$$\text{therefore } v = -\frac{\delta y^2}{2hb} + v(0). \tag{8}$$

This demonstrates that a quadrilateral element with linear shape functions can't satisfy the incompressibility constraint exactly in bending. Even if we ignore the impossibility of satisfying the incompressibility constraint pointwise throughout the element, it's also clear that the constraint can't be satisfied at the Gauss points for $2 \times 2$ integration because the sign of $\epsilon_{22}$ must change sign between the upper and lower rows of integration points. The sign change implies that $\epsilon_{22}$ must be at least linear in $y$, which therefore implies that $v$ must be at least quadratic in $y$, as in Equation 8.

There is one point where the incompressibility constraint is satisfied for the bending mode, namely the element centroid. The incompressibility constraint is, therefore, usually imposed at the element centroid in 4-node quadrilateral and 8-node brick elements.

# 3 Constraint Counting

A definitive way to determine whether locking will occur is the Babuska-Brezzi condition, but it is difficult to apply. Constraint counting, however, is an easy-to-apply, rough guide to determining whether an element formulation is likely to lock for incompressible problems. The basic idea is the ratio, $R$, of the equilibrium equations divided by the number of locations imposing the incompressibility condition should equal the ratio of the number of continuum equilibrium equations divided by the number of continuum constraints (which is simply 1 for the incompressibility condition).

## 3.1 Example: The Triangular Element

Assume that the number of nodes in Figure 3 is extended to $n + 1$ in each direction. The number of unconstrained nodes is therefore $n^2$, for $2n^2$ equilibrium equations, and the number of elements is $2n^2$, with each element contributing one volume constraint. Therefore, $R = 1$, implying that the number of degrees of freedom per volume constraint is too low. A value of 1 doesn't imply the rigid locking exhibited by this example; many formulations having $R = 1$ are simply overly stiff.

For the crossed triangle mesh, the number of nodes in the mesh is doubled to $4n^2$, as is the number of elements. Based on this superficial level of analysis, $R = 1$, and therefore this mesh should also lock up. However, it is well known that crossed triangle meshes don't lock.

A closer examination shows that that quadrilateral composed of four triangles has only three independent volume constraints, and therefore $R = 4/3$. To understand this, consider node $C$ in Figure 4, which belongs to triangles $\triangle DEC$ and $\triangle CEB$, and assume that nodes $D$, $E$ and $B$ are fixed. The areas of the triangles are conserved as long as $C$ moves parallel to the edges opposite it in each triangle, line segments $DE$ and $EB$. These two line segments are parallel, and therefore node $C$ can move along their common direction. In effect, the two triangles only impose one constraint on the motion of $C$. This situation is in contrast to the one in Figure 2, which also has two volume constraints, but which locks up completely.

The underlying reason that the crossed triangular mesh doesn't lock is each pair of adjacent elements has a common edge direction along one of the quadrilateral's diagonals. If node $E$ is moved so that $DE$ is no longer parallel to $EB$, then node $C$ can no longer move (and similarly for node $A$).

With the crossed triangles, each of the four vertex nodes ($A$, $B$, $C$, and $D$) of the quadrilateral has one degree of freedom. There are an additional three degrees of freedom associated with the rigid body modes, which exactly satisfy the volume constraints. The total number of degrees of freedom is therefore seven, and since the five nodes have a potential of ten degrees of freedom, the four volume constraints impose 10-7=3 actual constraints.

## 3.2 Example: Quadrilateral Elements

Again assuming that the number of nodes in each direction is $n + 1$, the number of elements and unconstrained nodes is $n^2$. Imposing the volume constraint at each integration point, the number of constraints is $4n^2$ for full integration. The ratio $R = 1/2$, and it would appear that the quadrilateral element should lock up like the triangular element in Figure 3. Since this isn't the case, constraint counting is at best a rough guide.

If the incompressibility constraint is imposed only at the center of the element, $R = 2$, the ideal ratio. In fact, this formulation works very well for incompressible problems whether the constraint is imposed via selective reduced integration, uniformly reduced integration, or the $\bar{\boldsymbol{B}}$ formulation.